

About the integrability of the Rapcsák equation

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May 20, 2015

Abstract

In [17] A. Rapcsák obtained necessary and sufficient conditions for the projective Finsler metrizable in terms of a second order partial differential equations. In this paper we investigate the integrability of the Rapcsák system, consisting of the Rapcsák equations and the homogeneity condition, by using the Spencer version of the Cartan-Kähler theorem. We also consider the extended Rapcsák system, where the first integrability conditions are included.

2010 MSC class: 49N45, 58E30, 53C60, 53C22

Keywords: sprays, projective metrizable, partial differential operator, formal integrability.

1 Introduction

Last year we celebrated the 100th years of the birth of András Rapcsák. He was one of the founder of the Finsler Geometry research school in Debrecen. Rapcsák's results are still relevant and up to date, as several recent citations show it. His most important results concern the projective Finsler metrizable problem, where one seeks for a Finsler metric whose geodesics are projectively equivalent to the solutions of a given second order homogeneous ordinary differential equations (SODE).

The projective Finsler metrizable problem can be considered as a particular case of the inverse problem of the calculus of variations. Rapcsák [17] obtained necessary and sufficient conditions for the projective Finsler metrizable in terms of a second order PDE system, called now *Rapcsák equations* [7, 20, 21]. The coordinate free formulations of these equations can be found in [11, 20]. Rapcsák's approach is simple and natural: one finds conditions directly on the Finsler function that one seeks for. Recently several new results appeared about the projective Finsler metrizable problem [3, 6, 7, 8, 12, 13]. We remark that, although in these papers the Rapcsák equation is usually mentioned, the new results were obtained by different approaches, for example by the so-called multiplier method, where one seeks for the existence of a variational multiplier matrix. In [7] the generalized Helmholtz system was considered and in [3] a system in terms of a semi-basic 1-form was investigated. In this paper, in the perspective of the projective metrizable

problem, we consider the *Rapcsák system*, which consists of the homogeneity equation (1) and a second order differential equation (9), called the *Rapcsák equation*. We investigate the integrability of the Rapcsák system by using the Spencer version of the Cartan-Kähler theorem.

The structure of the paper is as follows. In Section 2 we give a brief introduction to the Frölicher-Nijenhuis theory and to the canonical structures on the tangent bundle of a manifold. We also introduce the main structures one needs to discuss the geometry of a spray: connection, Jacobi endomorphism, curvature. We also recall the basic tool of the Cartan-Kähler theory.

In Section 3 we use the geometric setting presented in Section 2 to show that the Rapcsák system gives necessary and sufficient condition for the projective metrizability problem. Alternative proves can be found in [19, 20]. We discuss the integrability of the Rapcsák system by using conditions provided by Cartan-Kähler theorem. We conclude the chapter by showing that there is only one obstruction to the formal integrability of P_1 . This obstruction is expressed in terms of the nonlinear connection induced by the spray.

In Section 4 we investigate the formal integrability of the *extended Rapcsák system* composed by the Rapcsák system and its integrability conditions found in Section 3. We show that the obstruction to the integrability of this new system can be expressed in terms of the curvature tensor of the nonlinear connection induced by the spray. For some classes of sprays the curvature obstruction is identically satisfied: flat sprays, isotropic sprays, and sprays on 1- and 2-dimensional manifolds. For each of these classes the Weyl curvature of the spray is zero. Although, for some of these classes the projective metrizability problem has been discussed before by some authors, our approach in this work is different. This approach gives also the possibility to push forward the computation and consider sprays with non vanishing Weyl curvature. We remark however that in that case the computations are long and complex because the symbol of the correspondent differential operator may be not involutive. By computing the appropriate Spencer cohomology groups one can prove that it is not 2-acyclic either [15] which shows that higher order compatibility conditions arises. The analysis of this new system will be the subject of an other publication.

2 Preliminaries

Throughout this paper M will denote an n -dimensional smooth manifold. $C^\infty(M)$ denotes the ring of real-valued smooth functions, $\mathfrak{X}(M)$ is the $C^\infty(M)$ -module of vector fields on M , $\pi : TM \rightarrow M$ is the tangent bundle of M , $\mathcal{T}M = TM \setminus \{0\}$ is the slit tangent space. We will essentially work on the manifold TM and on its tangent space TTM . When there is no danger of confusion, TTM and T^*TM will simply be denoted by T and T^* , respectively. $VTM = \text{Ker } \pi_*$ is the vertical sub-bundle of T . We denote by $\Lambda^k(M)$, $S^k(M)$ and $\Psi^k(M)$ the $C^\infty(M)$ -modules of the skew-symmetric, symmetric and vector valued k -forms respectively. Similarly, we denote by $\Lambda_v^k(TM)$ and $\Psi_v^k(TM)$ the $C^\infty(TM)$ -modules of semi-basic k -forms and semi-basic vector valued k -forms.

The Frölicher-Nijenhuis theory provides a complete description of the derivations of $\Lambda(M)$ with the help of vector-valued differential forms, for details we refer to [9]. The i_* and the d_* type derivation associated to a vector valued l -form L will be denoted by i_L and d_L . They can be introduced in the following way: if $L \in \Psi^l(M)$, then

$$i_L \omega(X_1, \dots, X_l) = \omega(L(X_1, \dots, X_l)),$$

where $X_1, \dots, X_l \in \mathfrak{X}(M)$, $\omega \in \Lambda^1(M)$. Furthermore, d_L is the commutator of the deriva-

tions i_L and d , that is

$$d_L := [i_L, d] = i_L d - (-1)^{l-1} d i_L.$$

We remark, that for $X \in \mathfrak{X}(M)$ we have $d_X = \mathcal{L}_X$ the Lie derivative, and i_x is the substitution operator. The *Frölicher–Nijenhuis bracket* of $K \in \Psi^k(M)$ and $L \in \Psi^l(M)$ is the unique $[K, L] \in \Psi^{k+l}$ form, such that

$$[d_K, d_L] = d_{[K, L]}.$$

In the special case, when $K \in \Psi^1(M)$, $X, Y \in \mathfrak{X}(M)$ we have $[K, X] \in \Psi^1(M)$ defined as

$$[K, X](Y) = [KY, X] - K[Y, X].$$

Spray and associated geometric quantities

Let $J: TTM \rightarrow TTM$ be the *vertical endomorphism* and $C \in \mathfrak{X}(TM)$ the *Liouville vector field*. In an induced local coordinate system (x^i, y^i) on TM we have $J = dx^i \otimes \frac{\partial}{\partial y^i}$, and $C = y^i \frac{\partial}{\partial y^i}$. Euler's theorem for homogeneous functions implies that $L \in C^\infty(TM)$ is a 1-homogeneous function in the y variable if and only if

$$y^i \frac{\partial L}{\partial y^i} - L = 0. \quad (1)$$

The vertical endomorphism satisfies the following properties: $J^2 = 0$, $\text{Ker } J = \text{Im } J = VTM$ and $[J, C] = J$.

A *spray* is a vector field S on TM satisfying the relations $JS = C$ and $[C, S] = S$. The coordinate representation of a spray S takes the form

$$S = y^i \frac{\partial}{\partial x^i} + f^i(x, y) \frac{\partial}{\partial y^i},$$

where the functions $f^i(x, y)$ are homogeneous of degree 2 in y . The *geodesics of a spray* are curves $\gamma: I \rightarrow M$ such that $S \circ \dot{\gamma} = \ddot{\gamma}$. Locally, they are the solutions of the equations

$$\ddot{x}^i = f^i(x, \dot{x}), \quad i = 1, \dots, n. \quad (2)$$

Two sprays S and \tilde{S} are called *projective equivalent*, if their geodesics coincide up to an orientation preserving reparametrization. It is not difficult to show that S and \tilde{S} are projective equivalent if and only if they are related, by the formula:

$$\tilde{S} = S - 2\mathcal{P}C, \quad (3)$$

where $\mathcal{P} \in C^\infty(TM)$ is a 1-homogeneous function.

To every spray S a *connection* $\Gamma := [J, S]$ can be associated. We have $\Gamma^2 = \text{Id}$. The eigenspace of Γ corresponding to the eigenvalue -1 is the *vertical space* VTM , and the eigenspace corresponding to $+1$ is called the *horizontal space*. For any $x \in TM$ we have $T_x TM = H_x TM \oplus V_x TM$. The corresponding horizontal and vertical projectors associated to Γ are denoted by h and v . One has

$$h = \frac{1}{2}(\text{Id} + \Gamma), \quad v = \frac{1}{2}(\text{Id} - \Gamma). \quad (4)$$

The *curvature* $R = \frac{1}{2}[h, h]$ of the connection is the Nijenhuis torsion of the horizontal projection h . The *Jacobi endomorphism* (or Riemann curvature [21]) is defined as $\Phi = i_S R$. The Jacobi endomorphism determines the curvature by the formula $R = \frac{1}{3}[J, \Phi]$. The spray S is called *flat* if its Jacobi endomorphism has the form $\Phi = \lambda J$ and *isotropic*, if $\Phi = \lambda J - \alpha \otimes C$ with some $\lambda \in C^\infty(TM)$, $\alpha \in \Lambda_v^1(TM)$.

Finsler structure

A *Finsler function* on a manifold M is a continuous function $F: TM \rightarrow \mathbb{R}$, which is smooth and positive away from the zero section, homogeneous of degree 1, and strictly convex on each tangent space. The energy function $E: TM \rightarrow \mathbb{R}$ associated to a Finsler structure F is defined as $E := \frac{1}{2}F^2$. The

$$g_{ij} := \frac{\partial^2 E}{\partial y^i \partial y^j}$$

is positive definite at any points $(x, y) \in TM$. The pair (M, F) is called Finsler manifold. The geodesics on the Finsler manifold (M, F) are the solutions of the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i} = 0, \quad i = 1, \dots, n. \quad (5)$$

It is not difficult to see that for any function $E \in C^\infty(TM)$ the 1-form

$$\omega_E = i_S dd_J E + d\mathcal{L}_C E - dE, \quad (6)$$

is semi-basic, and its coordinate representation takes the form $\omega_E = \omega_i dx^i$ where the coefficients ω_i are the functions appearing in the left-hand side of the Euler-Lagrange equation (5). Therefore S corresponds to the geodesic equation of E if and only if the equation

$$\omega_E = 0 \quad (7)$$

holds. The spray S is called *Finsler metrizable*, if there exists a Finsler function such that for the corresponding energy function (7) holds, and S is *projective Finsler metrizable*, if it is projective equivalent to a *Finsler metrizable* spray.

Formal integrability

To investigate the integrability of the Rapcsák system we shall use Spencer's technique of formal integrability in the form explained in [10]. For a detailed account see [5]. We recall here the basic notions in order to fix the terminology.

Let B be a vector bundle over M . If s is a section of B , then $j_k(s)_x$ will denote the k^{th} order jet of s at the point $x \in M$. The bundle of k^{th} order jets of the sections of B is denoted by $J_k B$. In particular $J_k(\mathbb{R}_M)$ will denote the k^{th} order jets of real valued functions, that is the sections of the trivial line bundle. Let B_1 and B_2 be vector bundles over M and $P: \text{Sec}(B_1) \rightarrow \text{Sec}(B_2)$ a differential operator. An $s \in \text{Sec}(B_1)$ is a *solution* to P if $Ps \equiv 0$.

If P is a linear differential operator of order k , then a morphism $p_k(P): J_k(B_1) \rightarrow B_2$ can be associated to P . The l^{th} order prolongation $p_{k+l}(P): J_{k+l}(B_1) \rightarrow J_l(B_2)$ can be introduced in a natural way by taking the l^{th} order derivatives. $\text{Sol}_{k+l,x}(P) := \text{Ker } p_{k+l,x}(P)$ denotes the set of *formal solutions of order l at $x \in M$* . Obviously, we have

$$Ps \equiv 0 \quad \Rightarrow \quad j_{l,x}(s) \in \text{Sol}_{l,x}(P),$$

for every $l \geq k$ and $x \in M$. The differential operator P is called *formally integrable* if $\text{Sol}_l(P)$ is a vector bundle for all $l \geq k$, and the restriction $\bar{\pi}_{l,x}: \text{Sol}_{l+1,x}(P) \rightarrow \text{Sol}_{l,x}(P)$ of the natural projection is onto for every $l \geq k$. In that case any k^{th} order solution or *initial data* can be lifted into an infinite order solution. In the analytic case, formal integrability implies the existence of solutions for arbitrary initial data (see. [5], p. 397). To prove the

formal integrability, one can use the Cartan-Kähler theorem. To present it, we have to introduce some notations.

Let $\sigma_k(P)$ denote the symbol of P determined by the highest order terms of the operator. It can be interpreted as a map $\sigma_k(P): S^k T^*M \otimes B_1 \rightarrow B_2$. $\sigma_{k+l}(P): S^{k+l} T^*M \otimes B_1 \rightarrow S^l T^*M \otimes B_2$ denotes the symbol of the l^{th} order prolongation of P . If $\mathcal{E} = \{e_1 \dots e_n\}$ is a basis of $T_x M$, we set

$$\begin{aligned} g_{k,x}(P) &= \text{Ker } \sigma_{k,p}(P), \\ g_{k,x}(P)_{e_1 \dots e_j} &= \{A \in g_{k,p}(P) \mid i_{e_1} A = \dots = i_{e_j} A = 0\}, \quad j = 1, \dots, n, \end{aligned}$$

The basis \mathcal{E} is called *quasi-regular* if one has

$$\dim g_{k+1,x}(P) = \dim g_{k,x}(P) + \sum_{j=1}^n \dim g_{k,x}(P)_{e_1 \dots e_j}.$$

A symbol is called *involutive*¹ if there exists at any $x \in M$ a quasi-regular basis. The notion of involutivity allows us to check the formal integrability in a simple way by using the following

Theorem 2.1. [Cartan-Kähler]. *Let P be a k^{th} order linear partial differential operator. Suppose that P is regular, that is $\text{Sol}_{k+1}(P)$ is a vector bundle over $\text{Sol}_k(P)$. If the map $\bar{\pi}_k: \text{Sol}_{k+1}(P) \rightarrow \text{Sol}_k(P)$ is surjective and the symbol is involutive, then P is formally integrable.*

It can be shown that the condition of the existence of a quasi-regular basis can be replaced by a weaker condition. The obstructions to the higher order successive lift of the k^{th} order solution are contained in some of the cohomological groups of a certain complex called *Spencer complex*.

Using a classical result in homological algebra gives, the surjectivity of $\bar{\pi}_{k+1}$ can be verified in the following way [10]:

Proposition 2.2. *There exists a morphism $\varphi: \text{Sol}_k(P) \rightarrow \text{Coker}(\sigma_{k+1}(P))$, such that the sequence*

$$\text{Sol}_{k+1}(P) \xrightarrow{\bar{\pi}_k} \text{Sol}_k(P) \xrightarrow{\varphi} \text{Coker}(\sigma_{k+1}(P))$$

is exact. Therefore $\bar{\pi}_k$ is surjective if and only if $\varphi \equiv 0$.

Remark 2.3. The map φ is called *obstruction map* and $\text{Coker}(\sigma_{k+1}(P))$ is called *obstruction space*, because a k^{th} order solution $s \in \text{Sol}_k(P)$ can be prolonged into a $(k+1)^{\text{st}}$ order solution iff $\varphi(s) = 0$. In particular, if $\text{Coker}(\sigma_{k+1}(P)) = \{0\}$ then there is no obstruction to the prolongation.

In the practice the map φ and therefore the integrability conditions can be computed as follows:

Remark 2.4. Let be $\tau: T^* \otimes B_2 \rightarrow K$ a morphism such that $\text{Ker } \tau = \text{Im } \sigma_{k+1}(P)$. Then K is isomorphic to $\text{Coker}(\sigma_{k+1}(P))$. Moreover, if $s_{k,x} = j_k(s)_x$ is a k^{th} order solution, that is $(Ps)_x = 0$, then

$$\varphi(s_{k,x}) = \tau(\nabla(Ps))_x,$$

where ∇ is an arbitrary linear connection on the bundle B_2 .

¹In the works of Cartan, and more generally in the theory of exterior differential systems, "involutivity" means more than the existence of a quasi-regular basis and it refers to "integrability" (cf. [5], p.107, 140). Here we are following the terminology of Goldschmidt (cf. [5], p. 409).

Remark 2.5. Let (x^i) be a local coordinate system on M , (x^i, y^i) the associated coordinate system on TM in the neighborhood of $v \in TM$. If $j_k(F)_v \in J_k(\mathbb{R}_{TM})$ is a k th order jet of a real valued function F on TM we set

$$s_{i_1 \dots i_a \underline{i_{a+1} \dots i_l}} := \frac{\partial^l F}{\partial x^{i_1} \dots \partial x^{i_a} \partial y^{i_{a+1}} \dots \partial y^{i_l}}(v), \quad 1 \leq l \leq k. \quad (8)$$

Then $(s, s_i, s_{\underline{i}})$ and $(s, s_i, s_{\underline{i}}, s_{ij}, s_{\underline{ij}}, s_{\underline{ij}})$ give coordinate systems on $J_1(\mathbb{R}_{TM})$ and $J_2(\mathbb{R}_{TM})$ respectively.

3 Differential operator of the projective metrizable

In this section we derive the PDE system describing the necessary and sufficient condition for a spray to be projective Finsler metrizable. We have the following

Proposition 3.1. *A spray S is projective Finsler metrizable if and only if there exists a 1-homogeneous Lagrange function $\tilde{F}: TM \rightarrow \mathbb{R}$, such that $\frac{\partial^2 \tilde{F}^2}{\partial y^i \partial y^j}$ is positive definite on TM and*

$$i_S dd_J \tilde{F} = 0. \quad (9)$$

Proof. The spray S is projective Finsler metrizable if and only if there exists a Finsler metrizable spray \tilde{S} which is projective equivalent to S . Because of the projective equivalence, there exists a function \mathcal{P} , such that $\tilde{S} = S - 2\mathcal{P}C$. Let us denote by \tilde{F} the Finsler function associated to \tilde{S} . It is well known that \tilde{F} is invariant by the parallel translation associated to the connection $\tilde{\Gamma} = [J, \tilde{S}]$ and therefore we have $d_{\tilde{h}} \tilde{F} = 0$. Using the relation

$$\tilde{h} = h - \mathcal{P}J - d_J \mathcal{P} \otimes C$$

between the horizontal projectors ([4], chapter 4) and the 1-homogeneity of \tilde{F} , we get

$$0 = d_{\tilde{h}} \tilde{F} = d_h \tilde{F} - d_{\mathcal{P}J} \tilde{F} - d_J \mathcal{P} C \tilde{F} = d_h \tilde{F} - \mathcal{P} d_J \tilde{F} - \tilde{F} d_J \mathcal{P} = d_h \tilde{F} - d_J(\mathcal{P} \tilde{F}). \quad (10)$$

Substituting S in (10) and using $JS = C$ and the homogeneity of \tilde{F} and \mathcal{P} , we get

$$i_S d_{\tilde{h}} \tilde{F} = S \tilde{F} - C(\mathcal{P} \tilde{F}) = S \tilde{F} - 2\mathcal{P} \tilde{F} = 0$$

and we can find, that the projective factor is $\mathcal{P} = \frac{1}{2\tilde{F}} S \tilde{F}$. Replacing \mathcal{P} in (10) by the above expression we get

$$d_h \tilde{F} - d_J \left(\frac{1}{2\tilde{F}} (\tilde{F} d_S \tilde{F}) \right) = d_h \tilde{F} - \frac{1}{2} d_J (d_S \tilde{F}) = 0.$$

Using (4) and the relation $d_{[J, S]} = d_J d_S - d_S d_J$ we can obtain

$$\begin{aligned} 0 &= d_{\Gamma+I} \tilde{F} - d_J d_S \tilde{F} = d_{[J, S]} \tilde{F} + d \tilde{F} - d_J d_S \tilde{F} \\ &= -(i_S d + d i_S) d_J \tilde{F} + d \tilde{F} = -i_S dd_J \tilde{F} - dC \tilde{F} + d \tilde{F} = -i_S dd_J \tilde{F}. \quad \square \end{aligned}$$

We note, that a coordinate version of the above theorem was proved by A. Rapcsák in [17] and a coordinate free version was given in [11, 19, 20]. Here we presented a different proof.

Definition 3.2. Let S be a spray on M . The partial differential system composed by the equation (9) and the 1-homogeneity condition (1) is called the *Rapcsák system*.

According to Proposition 3.1 the projective metrizable leads to the investigation of the Rapcsák system.

Remark 3.3. The Rapcsák system is equivalent to the system composed by the Euler-Lagrange equations (5) and the 1-homogeneity condition (1).

We remark that the system composed by the Euler-Lagrange equations and the k -homogeneity condition for $k \neq 1$ can be reduced to a first order partial differential system which can be interpreted in terms of the holonomy distribution associated to the spray S . When $k = 2$, (this case corresponds to the Finsler metrizable problem) the computation can be found in [16]. The same reasoning can be applied for other value of k , $k \neq 1$. But this method cannot be used for the value $k = 1$. Nevertheless, in some special situations, the Rapcsák system can also be reduced to a first order PDE system. This is the case for example for the canonical spray of a Lie group, if one seeks for an invariant solution to the projective Finsler metrizable problem. In that case, the Rapcsák system can be reduced to a first order system, and one can show, that the invariant Riemann, Finsler and projective Finsler metrizable problems are equivalent [14].

Integrability conditions of the Rapcsák system

Let us consider the differential operator P_1 corresponding to the Rapcsák system:

$$P_1 = (P_S, P_C), \quad (11)$$

where

$$P_S: C^\infty(TM) \longrightarrow \text{Sec } T^*, \quad P_S(F) = i_S dd_J F, \quad (12)$$

$$P_C: C^\infty(TM) \longrightarrow C^\infty(TM), \quad P_C(F) = \mathcal{L}_C F - F. \quad (13)$$

From the local expression it is clear that P_C is a first and P_S is a second order differential operator. The associated morphisms are defined on the first and second order jet spaces respectively. Using the coordinate system (8) we get

$$\begin{aligned} p_1(P_C): J_1(\mathbb{R}_{TM}) &\longrightarrow \mathbb{R}, & j_1(F) &\longrightarrow y^i F_{\underline{i}} - F, \\ p_2(P_S): J_2(\mathbb{R}_{TM}) &\longrightarrow T^*, & j_2(F) &\longrightarrow (y^i F_{i\underline{j}} + f^i F_{\underline{i}j} - F_j) dx^i - (F_{\underline{i}} + y^j F_{\underline{i}j} - F_i) dy^i. \end{aligned}$$

The interesting feature of the Rapcsák system is that it is composed by differential operators of different orders. To find the integrability conditions of the system we consider the prolongation of the lower order equation. The morphism associated to this system is

$$p_2(P_1) = p_2(P_S) \times p_2(P_C): J_2(\mathbb{R}_{TM}) \longrightarrow T^* \times J_1(\mathbb{R}_{TM}).$$

Proposition 3.4. A 2^{nd} order solution of $P_1 = (P_S, P_C)$ at $x \in TM$ can be lifted into a 3^{rd} order solution, if and only if one has $i_\Gamma dd_J F = 0$ at x , where $\Gamma = [J, S]$ is the canonical nonlinear connection associated to S .

Proof. The symbols are defined by the highest order part of the operators. For P_C we find

$$\sigma_1(P_C): T^* \longrightarrow \mathbb{R}, \quad \sigma_1(P_C)A_1 = A_1(C).$$

The symbol of P_S and the prolongation of the symbol of P_C are

$$\begin{aligned}\sigma_2(P_C): S^2T^* &\longrightarrow T^*, & (\sigma_2(P_C)A_2)(X) &= A_2(X, C), \\ \sigma_2(P_S): S^2T^* &\longrightarrow T^*, & (\sigma_2(P_S)A_2)(X) &= A_2(S, JX) - A_2(X, C),\end{aligned}$$

for every $X \in T$, $A_1 \in T^*$, $A_2 \in S^2T^*$. The prolongations of the symbols at third order level are

$$\begin{aligned}\sigma_3(P_C): S^3T^* &\longrightarrow T^* \otimes S^2T^*, & (\sigma_3(P_C)A_3)(X, Y) &= A_3(X, Y, C), \\ \sigma_3(P_S): S^3T^* &\longrightarrow T^* \otimes T^*, & (\sigma_3(P_S)A_3)(X, Y) &= A_3(X, S, JY) - A_3(X, Y, C),\end{aligned}$$

where $X, Y \in T$, $A_3 \in S^3T^*$ and we have

$$\sigma_3(P_1) = (\sigma_3(P_S), \sigma_3(P_C)): S^3T^* \longrightarrow (T^* \otimes T^*) \times S^2T^*.$$

Let us consider the map $\tau_1 := (\tau_S^1, \tau_S^2, \tau_{SC}^1, \tau_{SC}^2)$ where

$$\tau_S^1(B_S, B_C)(X, Y) = B_S(JX, hY) - B_S(hY, JX) - B_S(JY, hX) + B_S(hX, JY), \quad (14)$$

$$\tau_S^2(B_S, B_C)(X) = B_S(X, S), \quad (15)$$

$$\tau_{SC}^1(B_S, B_C)(X, Y) = B_S(X, JY) + B_C(X, JY), \quad (16)$$

$$\tau_{SC}^2(B_S, B_C)(X, Y) = B_S(C, hX) - B_C(S, JX) + B_C(hX, C), \quad (17)$$

for $B_S \in T^* \otimes T^*$, $B_C \in S^2T^*$, $X, Y \in T$.

Lemma 3.5. *We have $\text{Im } \sigma_3(P_1) = \text{Ker } \tau_1$ that is, if we denote $K_1 = \text{Im } \tau_1$ then the sequence*

$$S^3T^* \xrightarrow{\sigma_3(P_1)} (T^* \otimes T^*) \times S^2T^* \xrightarrow{\tau_1} K_1 \rightarrow 0 \quad (18)$$

is exact.

Proof. By the construction, we have to check the exactness in the second term. It is easy to compute that $\sigma_3(P_1) \circ \tau_1 = 0$ and therefore $\text{Im } \sigma_3(P_1) \subset \text{Ker } \tau_1$. Let us compute $\dim \text{Ker } \sigma_3(P_1)$. We consider the basis

$$\mathcal{B} := \{h_1, \dots, h_n, v_1, \dots, v_n\} \subset T_x, \quad (19)$$

where h_i are horizontal, $h_n = S$, $Jh_i = v_i$, $i = 1, \dots, n$ (and therefore $v_n = C$). In the sequel we denote the components of a symmetric tensor $A \in S^kT^*$ with respect to (19) as

$$A_{i_1 \dots i_j \underline{i_{j+1} \dots i_k}} := A(h_{i_1}, \dots, h_{i_j}, v_{i_{j+1}}, \dots, v_{i_k}). \quad (20)$$

It is clear that $\text{Ker } \sigma_3(P_1) = \text{Ker } \sigma_3(P_S) \cap \text{Ker } \sigma_3(P_C)$. The symmetric tensor $A \in S^3T^*$ is in $\text{Ker } \sigma_3(P_C)$ if

$$A_{ijn} = A_{ij\underline{n}} = A_{\underline{ijn}} = 0, \quad (21)$$

and $A \in S^3T^*$ is an element of $\text{Ker } \sigma_3(P_S)$ if

$$\sigma_3(P_S)(A)(h_i, h_j) = A(h_i, h_n, v_j) - A(h_i, h_j, v_n) = A_{in\underline{j}} - A_{ij\underline{n}} = 0, \quad (22)$$

$$\sigma_3(P_S)(A)(h_i, v_j) = -A(h_i, v_j, v_n) = -A_{ij\underline{n}} = 0, \quad (23)$$

$$\sigma_3(P_S)(A)(v_i, h_j) = A(v_i, h_n, v_j) - A(v_i, h_j, v_n) = A_{\underline{ijn}} - A_{ij\underline{n}} = 0, \quad (24)$$

$$\sigma_3(P_S)(A)(v_i, v_j) = -A(v_i, v_j, v_n) = -A_{\underline{ijn}} = 0, \quad (25)$$

for $i, j = 1, \dots, n$. Taking into account the symmetry of A we have $2\frac{n(n+1)}{2} + n^2$ independent equations in (21). Moreover, counting the independent equations in (22)-(25) we get that (23) and (25) trivially hold because of (21). From (22) we have only $n^2 - n$ independent equations because for $j = n$ the equations are trivially satisfied, and from (24) we have $\frac{n(n-1)}{2}$ independent equations because again, for $j = n$ they are trivially satisfied. Consequently, we have $2\frac{n(n+1)}{2} + 2n^2 - n + \frac{n(n-1)}{2} = \frac{7n^2 - n}{2}$ independent equations in the system (21)-(25). Therefore we get

$$\dim(g_3(P_1)) = \dim \text{Ker } \sigma_3(P_1) = \dim S^3 T^* - \frac{7n^2 - n}{2} = \frac{8n^3 - 9n^2 + 7n}{6} \quad (26)$$

and

$$\text{rank } \sigma_3(P_1) = \frac{7n^2 - n}{2}. \quad (27)$$

On the other hand, let us compute $\dim \text{Ker } \tau_1$. The pivot terms for the equation $\tau_S^1 = 0$ are $B_S(v_i, h_j)$, $i < j < n$. Furthermore, $B_S(v_i, h_n)$, $B_S(h_i, h_n)$, $i = 1, \dots, n$ are pivot terms for $\tau_S^2 = 0$. Therefore the number of independent equations for $\text{Ker } \tau_S^1$ and $\text{Ker } \tau_S^2$ are $\frac{(n-1)(n-2)}{2}$ and $2n$, respectively. Moreover, the pivot terms for the equations $\tau_{SC}^1 = 0$ and $\tau_{SC}^2 = 0$ are $B_S(h_i, v_j)$, $B_S(v_i, v_j)$, $i, j = 1, \dots, n$, and $B_S(v_n, h_i)$, $i = 1, \dots, n-1$, giving in addition $2n^2 + n - 1$ independent equations.

$$\dim \text{Ker } \tau_1 = \dim S^2 T^* + \dim(T^* \otimes T^*) - \left[\frac{(n-1)(n-2)}{2} + 2n^2 + 3n - 1 \right] = \frac{7n^2 - n}{2}. \quad (28)$$

Comparing (27) and (28) we get $\text{Im } \sigma_3(P_1) = \text{Ker } \tau_1$. \square

Proof of Proposition 3.4. The morphisms, the symbols and the obstruction map associated to the Rapcsák system can be represented in the following commutative diagram:

$$\begin{array}{ccccccc} g_3(P_1) & \longrightarrow & S^3 T^* & \xrightarrow{\sigma_3(P_1)} & (T^* \otimes T^*) \times S^2 T^* & \xrightarrow{\tau_1} & K_1 \longrightarrow 0 \\ \downarrow & & \downarrow \epsilon & & \downarrow \epsilon & & \\ \text{Sol}_3(P_1) & \xrightarrow{i} & J_3(\mathbb{R}_{TM}) & \xrightarrow{p_3(P_1)} & J_1(T^*) \times J_2(\mathbb{R}_{TM}) & & \\ \downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow \pi_0 \times \pi_1 & & \\ \text{Sol}_2(P_1) & \xrightarrow{i} & J_2(\mathbb{R}_{TM}) & \xrightarrow{p_2(P_1)} & T^* \times J_1(\mathbb{R}_{TM}) & & \end{array}$$

Let $s = j_2(F)_x \in \text{Sol}_{2,x}(P_1)$ be a second order solution of P_1 at x , that is

$$(i_S \text{dd}_J F)_x = 0, \quad (\mathcal{L}_C F - F)_x = 0, \quad (\nabla(\mathcal{L}_C F - F))_x = 0. \quad (29)$$

The integrability condition can be computed in terms of $\tau_1 = (\tau_S^1, \tau_S^2, \tau_{SC}^1, \tau_{SC}^2)$. According to Remark 2.4, s can be lifted into a third order solution if and only if $\varphi(s) = 0$, where $\varphi(s) = (\tau_1 \nabla P_1(F))_x$. Computing $\varphi(s)$ we find that

1. using the notation $\omega := i_S \text{dd}_J F$ we have $\omega_x = 0$ from (29) and

$$\begin{aligned} \tau_S^1(\nabla(P_1 F))_x(X, Y) &= \nabla \omega(JX, hY) - \nabla \omega(hY, JX) - \nabla \omega(JY, hX) + \nabla \omega(hX, JY) \\ &= JX \omega(hY) - hY \omega(JX) - JY \omega(hX) + hX \omega(JY) = i_J d\omega(hX, hY). \end{aligned}$$

Moreover, $di_S = -i_S d + d_S$, $i_J d_S = i_{[J,S]} + d_S i_J$ and $d_J d_J = 0$, we obtain that

$$\begin{aligned} i_J d\omega(hX, hY)_x &= (i_J d_S dd_J F - i_J i_S ddd_J F)_x(hX, hY) \\ &= (i_{[J,S]} dd_J F + d_S i_J dd_J F)_x(hX, hY) = (i_\Gamma dd_J F)_x(hX, hY). \end{aligned}$$

$$2. \tau_S^2(\nabla(P_1 F))_x = (\nabla\omega)_x(X, S) = X_x\omega(S) = X_x dd_J(S, S) = 0.$$

3. Using the identity $J[JX, S] = JX$ we have

$$\begin{aligned} \tau_{SC}^1(\nabla(P_1 F))_x &= X_x(i_S dd_J F(JY)) + X_x(JY(CF - F)) \\ &= X_x(-JY d_J F(S) - d_J F([S, JY])) + X_x(JYCF - JYF) \\ &= -X_x(J[S, JY]F) - X_x(JYF) = X_x(JYF) - X_x(JYF) = 0. \end{aligned}$$

4. We have $dd_J(CF - F)(S, hX) = S(JX(CF - F)) - hX(C(CF - F))$. Then

$$\begin{aligned} \tau_{SC}^2(\nabla(P_1 F))_x &= C(i_S dd_J F(hX)) - S(JX(CF - F)) + hX(C(CF - F)) \\ &= d_C dd_J F(S, hX) - dd_J d_C F(S, hX) + dd_J F(S, hX). \end{aligned}$$

Since $d_J d_C - d_C d_J = d_{[J,C]} = d_J$ it follows that

$$d_C dd_J F - dd_J d_C F + dd_J F = d_C dd_J F - dd_C d_J F = d_C dd_J F - d_C dd_J F = 0.$$

From the above computation it follows that $\varphi(s) = (\tau_1 \nabla P_1(F))_x = (i_\Gamma dd_J F_x, 0, 0, 0)$ and therefore the only condition to prolong a second order solution into a third order solution is given by the equation $(i_\Gamma dd_J F)_x = 0$ as Proposition 3.4 stated. ■

Proposition 3.6. *The symbol of $P_1 = (P_S, P_C)$ is involutive.*

Proof. Let us consider the basis \mathcal{B} introduced in (19). Using the notation (20) we have

$$\begin{aligned} g_2(P_1) = \text{Ker } \sigma_2(P_1) &= \{A \in S^2 T^* | A(X, C) = 0, A(S, JX) = A(X, C)\} \\ &= \left\{A \in S^2 T^* | A_{ij} = A_{ji}, A_{\underline{ij}} = A_{\underline{ji}}, A_{\underline{in}} = A_{n\underline{i}} = A_{\underline{in}} = A_{n\underline{i}} = 0\right\} \end{aligned}$$

and therefore

$$\dim(g_2(P_1)) = \frac{n(n+1)}{2} + (n-1)^2 + \frac{n(n-1)}{2} = n^2 + (n-1)^2. \quad (30)$$

Let us consider now the basis $\mathcal{E} = \{e_i\}_{i=1\dots 2n}$, where

$$\mathcal{E} = \left\{ \underbrace{h_1}_{e_1}, \dots, \underbrace{h_{n-1}}_{e_{n-1}}, \underbrace{h_n + v_1 + \dots + v_n}_{e_n}, \underbrace{v_1}_{e_{n+1}}, \dots, \underbrace{v_n}_{e_{2n}} \right\}. \quad (31)$$

Denoting the coefficients of $A \in S^2 T^*$ with respect to \mathcal{E} by \tilde{A}_{ij} , we have

$$\begin{aligned} g_2(P_1)_{e_1 \dots e_k} &= \{A \in S^2 T^* | i_{e_1} A = 0, \dots, i_{e_k} A = 0\} \\ &= \left\{A \in S^2 T^* | \tilde{A}_{ij} = \tilde{A}_{ji}, \tilde{A}_{\underline{ij}} = \tilde{A}_{\underline{ji}}, \tilde{A}_{\underline{in}} = \tilde{A}_{n\underline{i}} = \tilde{A}_{\underline{in}} = \tilde{A}_{n\underline{i}} = 0, \tilde{A}_{lj} = 0, \tilde{A}_{\underline{l}} = 0, l \leq k\right\} \end{aligned}$$

therefore

$$\dim(g_2(P_1))_{e_1 \dots e_k} = \begin{cases} \frac{(n-k)(n-k+1)}{2} + (n-k)(n-1) + \frac{(n-2)(n-1)}{2}, & \text{if } k \leq n, \\ \frac{(n-2-k)(n-k-1)}{2}, & \text{if } k > n, \end{cases}$$

and hence

$$\begin{aligned} \dim g_2(P_1) + \sum_{k=1}^{2n} \dim g_2(P_1)_{e_1 \dots e_k} &= n^2 + (n-1)^2 + \sum_{k=1}^n \left(\frac{(n-k)(n-k+1)}{2} + (n-k)(n-1) \right) \\ &+ \frac{n(n-2)(n-1)}{2} + \sum_{k=1}^n \frac{(n-2-k)(n-k-1)}{2} = \frac{8n^3 - 9n^2 + 7n}{6} \stackrel{(26)}{=} \dim g_3(P_1), \end{aligned}$$

which shows that the basis (31) is quasi-regular, and the symbol of P_1 is involutive. \square

Remark 3.7. Proposition 3.4 and 3.6 shows that the conditions of Theorem 2.1 are fulfilled if and only if for any initial data $j_2(F)_x$ of P_1 we have also $i_\Gamma dd_J F = 0$. This is true if $\dim M = 1$. However, when $\dim M \geq 2$, this condition does not satisfied by every second order solution, that is not every second order solution can be lifted into a third order solution. Since the set of initial data is too large (containing some which cannot be prolonged into a higher order solution) we have to reduce it by adding the compatibility condition into the system. This leads us to consider the operator (P_S, P_C, P_Γ) where P_Γ is a second order operator defined as

$$P_\Gamma : C^\infty(TM) \rightarrow \text{Sec}(\Lambda^2 T_v^*), \quad P_\Gamma F := i_\Gamma dd_J F.$$

Remark 3.8. If S is a spray and F is a 1-homogeneous Lagrangian, then we have

$$P_S F(X) = i_S dd_J F(X) = dd_J F(S, hX) = \frac{1}{2} i_\Gamma dd_J F(S, hX) = P_\Gamma F(S, hX)$$

for every $X \in T$. Consequently, if F is a solution of P_Γ then it is also a solution of P_S , that is P_Γ contains in particular the equations of P_S . That lead us to drop from the system P_S and consider the *extended Rapcsák* system as:

$$P_2 = (P_\Gamma, P_C). \tag{32}$$

It is clear that a function is a solution to the Rapcsák system if and only if it is a solution of the extended Rapcsák system.

4 Integrability condition of the extended Rapcsák system

In this chapter we investigate the integrability of the extended Rapcsák system $P_2 = (P_\Gamma, P_C)$. Our method is similar to the one we used in Chapter 3.

Proposition 4.1. *A 2^{nd} order solution $s = j_2(F)_x$ of the system $P_2 = (P_\Gamma, P_C)$ at $x \in TM$ can be prolonged into a 3^{rd} order solution, if and only if $(i_R dd_J F)_x = 0$.*

Proof. The symbol of the operator P_Γ and its first prolongation are

$$\begin{aligned} \sigma_2(P_\Gamma) : S^2 T^* &\rightarrow \Lambda^2 T_v^*, & (\sigma_2(P_\Gamma) A_2)(Y, Z) &= 2(A_2(hY, JZ) - A_2(hZ, JY)), \\ \sigma_3(P_\Gamma) : S^3 T^* &\rightarrow T^* \otimes \Lambda^2 T_v^*, & (\sigma_3(P_\Gamma) A_3)(X, Y, Z) &= 2(A_3(X, hY, JZ) - A_3(X, hZ, JY)), \end{aligned}$$

where $X, Y, Z \in T$, $A_2 \in S^2 T^*$, $A_3 \in S^3 T^*$. Let us consider the map

$$\tau_2 := (\tau_\Gamma^1, \tau_\Gamma^2, \tau_{\Gamma C}) \tag{33}$$

defined on $(T^* \otimes \Lambda^2 T_v^*) \times S^2 T^*$ with

$$\tau_\Gamma^1(B_\Gamma, B_C)(X, Y, Z) = B_\Gamma(hX, Y, Z) + B_\Gamma(hY, Z, X) + B_\Gamma(hZ, X, Y), \quad (34)$$

$$\tau_\Gamma^2(B_\Gamma, B_C)(X, Y, Z) = B_\Gamma(JX, Y, Z) + B_\Gamma(JY, Z, X) + B_\Gamma(JZ, X, Y), \quad (35)$$

$$\tau_{TC}(B_\Gamma, B_C)(X, Y) = \frac{1}{2}B_\Gamma(C, X, Y) - B_C(hX, JY) + B_C(hY, JX), \quad (36)$$

where $B_\Gamma \in T^* \otimes \Lambda^2 T_v^*$, $B_C \in S^2 T^*$, $X, Y, Z \in T$. We have the following

Lemma 4.2. *Let K_2 be the image of τ_2 . Then the sequence*

$$S^3 T^* \xrightarrow{\sigma_3(P_2)} (T^* \otimes \Lambda^2 T_v^*) \times S^2 T^* \xrightarrow{\tau_2} K_2 \longrightarrow 0 \quad (37)$$

is exact.

Proof. A simple computation shows that $\sigma_3(P_2) \circ \tau_2 = 0$, and therefore $\text{Im } \sigma_3(P_2) \subset \text{Ker } \tau_2$. Let us compute the rank of $\sigma_3(P_2)$. By using the basis (19) and the notation (20), a symmetric tensor $A \in S^3 T^*$ is an element of $\text{Ker } \sigma_3(P_2)$ if in addition of the relations describing the symmetry properties, the equations (21) and the equations

$$A_{i\bar{j}\bar{k}} = A_{ik\bar{j}}, \quad A_{i\bar{j}\bar{k}} = A_{ik\bar{j}}, \quad i, j, k = 1, \dots, n, \quad (38)$$

hold. We obtain from (38) that all of the blocks (20) are totally symmetric, and $A_{i\bar{j}\bar{n}} = A_{i\bar{j}\bar{n}} = 0$. That way there are $\frac{n(n+1)(n+2)}{6}$ free components in the block $A_{i\bar{j}\bar{k}}$ and $\frac{(n-1)n(n+1)}{6}$ free components to choose in each of the blocks $A_{i\bar{j}\bar{k}}$, $A_{i\bar{j}\bar{k}}$ and $A_{i\bar{j}\bar{k}}$. That is

$$\dim(g_3(P_2)) = \frac{n(n+1)(n+2)}{6} + 3 \frac{(n-1)n(n+1)}{6} = \frac{4n^3 + 3n^2 - n}{6}, \quad (39)$$

and

$$\text{rank } \sigma_3(P_2) = \dim S^3 T^* - \dim(g_3(P_2)) = \frac{4n^3 + 9n^2 + 5n}{6}. \quad (40)$$

On the other hand, considering the equations of $\text{Ker } \tau_2$, we can find that the pivot terms for $\tau_\Gamma^1 = 0$ and for $\tau_\Gamma^2 = 0$ are $B_\Gamma(h_i, h_j, h_k)$ and $B_\Gamma(v_i, h_j, h_k)$, $i < j < k$, respectively. Therefore each of them give $\binom{n}{3}$ independent equations. Furthermore $B_\Gamma(v_n, h_i, h_j)$, $i < j$, $i, j = 1 \dots n$, are pivot terms for $\tau_{TC} = 0$ which gives $\frac{n(n-1)}{2}$ independent equations. Hence

$$\dim \text{Ker } \tau_2 = \dim S^2 T^* + \dim(T^* \otimes \Lambda^2 T_v^*) - 2 \binom{n}{3} - \frac{n(n-1)}{2} = \frac{4n^3 + 9n^2 + 5n}{6}. \quad (41)$$

Comparing the dimensions (40) and (41) one can find that $\text{rank } \sigma_3(P_2) = \dim \text{Ker } \tau_2$ and the sequence (37) is exact. \square

Let us turn our attention to the proof of Proposition 4.1. We have the following commutative diagram:

$$\begin{array}{ccccccc} g_3(P_2) & \longrightarrow & S^3 T^* & \xrightarrow{\sigma_3(P_2)} & (T^* \otimes \Lambda^2 T_v^*) \times S^2 T^* & \xrightarrow{\tau_2} & K_2 \longrightarrow 0 \\ \downarrow & & \downarrow \epsilon & & \downarrow \epsilon & & \\ \text{Sol}_3(P_2) & \xrightarrow{i} & J_3(\mathbb{R}_{TM}) & \xrightarrow{p_3(P_2)} & J_1(\Lambda^2 T_v^*) \times J_2(\mathbb{R}_{TM}) & & \\ \downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow & & \\ \text{Sol}_2(P_2) & \xrightarrow{i} & J_2(\mathbb{R}_{TM}) & \xrightarrow{p_2(P_2)} & \Lambda^2 T_v^* \times J_1(\mathbb{R}_{TM}) & & \end{array}$$

Let $s = j_2(F)_x$ be a second order solution of P_2 at a point x , that is $(P_2F)_x = 0$. We have

$$(i_\Gamma dd_J F)_x = 0, \quad (\mathcal{L}_C F - F)_x = 0, \quad (\nabla(\mathcal{L}_C F - F))_x = 0. \quad (42)$$

The integrability condition can be computed with the help of the map τ_2 (see Proposition 2.2 and 2.4). Indeed, $s \in \text{Sol}_{2,x}(P_2)$ can be prolonged into a third order solution if and only if $\varphi(s) = 0$, where $\varphi(s) = (\tau_2 \nabla P_2(F))_x$. Let us introduce the notation $\Omega = dd_J F$. Using the component maps of τ_2 introduced in (33) one can find

1. $\tau_\Gamma^1(\nabla(P_2F))_x = d_h(i_\Gamma dd_J F)_x = (d_h i_{2h-I} dd_J F)_x = (2(d_h i_h dd_J F - d_h dd_J F))_x = (2d_h(i_h d - d i_h) d_J F)_x = (2d_h d_h d_J F)_x = (d_R d_J F)_x = (i_R \Omega)_x$.
2. $\tau_\Gamma^2(\nabla(P_2F))_x = d_J(i_\Gamma \Omega)_x \stackrel{(4)}{=} (d_J(i_{2h-I} \Omega))_x = (2d_J i_h dd_J F - 2d_J dd_J F)_x = (-2d_J i_h d_J dF - 2i_J ddd_J F + 2di_J dd_J F)_x = -(2d_J(d_J i_h dF + d_J dF))_x = 0$ where we used $[d, d_J] = 0$, $[i_h, d_J] = d_{Jh} - i_{[h, J]}$ and $[J, h] = 0$.
3. $\tau_{\Gamma C}(\nabla P_2(F))_x(X, Y) = \frac{1}{2} \nabla i_\Gamma \Omega(C, X, Y) - \nabla P_C F(hX, JY) + \nabla P_C F(hY, JX) = \frac{1}{2} d_C i_\Gamma \Omega(hX, hY) - \frac{1}{2} i_\Gamma d_C dd_J F(hX, hY) = \frac{1}{2} d_{[C, \Gamma]} \Omega(hX, hY) \stackrel{[C, \Gamma]=0}{=} 0$.

The above computation shows that $\varphi(s) = \tau_2(\nabla P_2(F))_x = (i_R \Omega_x, 0, 0)$ which proves Proposition 4.1. \blacksquare

Proposition 4.3. *The symbol of P_2 is involutive.*

Proof. We consider the basis (19) and use the notation (20). We have

$$\begin{aligned} g_2(P_2) = \text{Ker } \sigma_2(P_1) &= \{A \in S^2 T^* \mid A(X, C) = 0, A(hX, JY) = A(hY, JX)\} \\ &= \left\{A \in S^2 T^* \mid A_{ij} = A_{ji}, A_{i\bar{j}} = A_{j\bar{i}}, A_{i\bar{j}} = A_{j\bar{i}}, A_{i\bar{n}} = 0, A_{n\bar{i}} = 0\right\}. \end{aligned}$$

Therefore $\dim(g_2(P_2)) = \frac{n(n+1)}{2} + 2\frac{(n-1)n}{2}$. Let us consider the basis $\hat{\mathcal{E}} = \{\hat{e}_i\}_{i=1}^{2n}$, where

$$\begin{aligned} \hat{e}_i &= h_i + i v_i, & i &= 1, \dots, n-1, \\ \hat{e}_n &= h_n + v_1 + \dots + v_n, \\ \hat{e}_{i+n} &= v_i, & i &= 1, \dots, n. \end{aligned}$$

In the new basis the components of the block $\hat{A}_{ij} = A_{ij}$ can be expressed as a combination of the components $\hat{A}_{i\bar{j}}$ as follows: when $i \neq j$, then $\hat{A}_{ij} = \frac{1}{i-j}(\hat{A}_{i\bar{j}} - \hat{A}_{j\bar{i}})$, and if $i = j$ we have $\hat{A}_{jj} = \hat{A}_{n\bar{j}} - \sum_{k \neq j} \frac{1}{k-j}(\hat{A}_{j\bar{k}} - \hat{A}_{k\bar{j}})$. Then,

$$\dim(g_2(P_2))_{\hat{e}_1 \dots \hat{e}_k} = \begin{cases} \frac{(n-k+1)(n-k)}{2} + (n-1)(n-k), & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

and therefore

$$\begin{aligned} \dim(g_2(P_2)) + \sum_{k=1}^{2n} \dim(g_2(P_2))_{\hat{e}_1 \dots \hat{e}_k} &= \\ &= \frac{n(n+1)}{2} + 2\frac{(n-1)n}{2} + \sum_{k=1}^n \left(\frac{(n-k+1)(n-k)}{2} + (n-1)(n-k) \right) \stackrel{(39)}{=} \dim(g_3(P_2)), \end{aligned}$$

and the basis $\hat{\mathcal{E}}$ is quasi-regular. \square

From Proposition 4.1 and 4.3, using the Cartan–Kähler theorem, we get that the integrability condition of the extended Rapcsák equation can be given in terms of the curvature tensor and we have the following

Theorem 4.4. *Let S be a spray on a manifold M . If*

1. $\dim M = 2$,
2. *the spray S is flat,*
3. *the spray S is of isotropic curvature,*

then the extended Rapcsák equation is formally integrable.

Proof. To prove the formal integrability one has to show that $\bar{\pi}_l: \text{Sol}_{l+1}(P) \rightarrow \text{Sol}_l(P)$ maps are surjective for any $l \geq 2$. Let $s = j_2(F)_x$ be a second order solution of the system (P_Γ, P_C) at $x \in \mathcal{T}M$. According to Proposition 4.1 it can be prolonged into a 3rd order solution if and only if $(i_R dd_J F)_x = 0$ holds.

1. If $\dim M = 2$, then the space of semi-basic 3-forms is trivial, that is $\Lambda_v^3(TM) = \{0\}$. Therefore $i_R dd_J F = 0$.
2. If S is flat, that is $\Phi = \lambda J$, then $R = d_J \lambda \wedge J$. Using the integrability of the vertical distribution we get: $i_R dd_J F = d_R d_J F = d_J \lambda d_J^2 F + d_{d_J \lambda} \wedge i_J d_J F = 0$.
3. If S is of isotropic curvature, then R takes the form $R = \alpha \wedge J + \beta \otimes C$, where $\alpha \in \Lambda_v^1(TM)$, $\beta \in \Lambda_v^2(TM)$. Then

$$i_R dd_J F = i_{\alpha \wedge J + \beta \otimes C} dd_J F = \alpha \wedge i_J dd_J F + \beta \wedge i_C dd_J F = 0.$$

The above computation shows that in the above cases all 2nd order solutions can be prolonged into a 3rd order solution. Moreover, as Theorem 4.2 shows, the symbol is involutive. Therefore, according Cartan–Kähler theorem, the operator P_2 is formally integrable. \square

Corollary 4.5. *Let S be a analytic spray on an analytic manifold M . If M is 2-dimensional, or the spray S is flat, resp. of isotropic curvature, then S is locally projectively Finsler metrizable.*

Indeed, in the analytic context, formal integrability implies the existence of solutions for all the initial data.

The integrability condition $i_R dd_J F = 0$ also appeared in [1]. It can be shown (c.f. [7]) that this integrability condition is equivalent to the equation $i_\Phi dd_J F = 0$ or $i_W dd_J F = 0$, where Φ is the Jacobi endomorphism and W is the Weyl tensor associated to S .

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